

THE DOUBLE OF REPRESENTATIONS OF COHOMOLOGICAL HALL ALGEBRA FOR A_1 -QUIVER

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ABSTRACT. We compute two representations of COHA for A_1 -quiver. The two untwisted representations can be combined into a representation of D_{n+1} Lie algebra. The untwisted increasing representation and the twisted decreasing representation can be combined into a representation of a finite Clifford algebra.

1. INTRODUCTION

The aim of this paper is to define and discuss two representations of the Cohomological Hall algebras, and combine them into a single representation of the algebra which is called “full” (or “double”) COHA in [9].

Cohomological Hall algebra (COHA for short) was introduced in [5]. The definition is similar to the definition of conventional Hall algebra (see e.g. [8]) or its motivic version (see e.g. [4]). Instead of spaces of constructible functions on the stack of objects of an abelian category, one considers cohomology groups of the stacks. The product is defined through the pullback-pushforward construction. Details can be found in [5].

By analogy with conventional Hall algebra of a quiver, which gives the “positive” part of a quantization of the corresponding Lie algebra, one may want to define the “double” COHA, for which the one defined in [5] would be a “positive part”. Following the discussion in [9], we study the double of representations of COHA, and hope to find the double of COHA through its representations.

This paper focuses on A_1 -quiver. Stable framed representations of the quiver are used to produce two representations of COHA. Since the moduli spaces of stable framed representations of A_1 -quiver are Grassmannians, we actually define two representations on the cohomology of Grassmannians. We show that the operators from these two representations form D_{n+1} -Lie algebra. We also make a modification to the decreasing representation and form a twisted decreasing representation. The operators from untwisted increasing operators and twisted decreasing operators form a finite Clifford algebra. These confirm the conjecture from [9] that the double of A_1 -COHA is the infinite Clifford algebra.

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2. TWO GEOMETRIC REPRESENTATIONS OF A_1 -COHA

2.1. COHA. Let Q be a quiver with N vertices. Given a dimension vector $\gamma = (\gamma_i)_{i=1}^N$, M_γ is the space of complex representations with fixed underlying vector space $\bigoplus_{i=1}^N \mathbb{C}^{\gamma_i}$ of dimension vector γ , and $G_\gamma = \prod_{i=1}^N GL_{\gamma_i}(\mathbb{C})$ is the associated gauge group. $[M_\gamma/G_\gamma]$ is the stack of representations of Q with fixed dimension vector γ . As a vector space, COHA of Q is defined to be $\mathcal{H} := \bigoplus_\gamma \mathcal{H}_\gamma := \bigoplus_\gamma H^*([M_\gamma/G_\gamma]) := \bigoplus_\gamma H_{G_\gamma}^*(M_\gamma)$. Here by equivariant cohomology of a complex algebraic variety M_γ acted by a complex algebraic group G_γ we mean the usual (Betti) cohomology with coefficients in \mathbb{Q} of the bundle $EG_\gamma \times_{G_\gamma} M_\gamma$ associated to the universal G_γ -bundle $EG_\gamma \rightarrow BG_\gamma$ over the classifying space of G_γ . The product $*$: $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ is defined by means of the pullback-pushforward construction in [5].

2.2. A_1 -COHA. Let Q be A_1 . $N = 1$. Since there is only one representation with fixed underlying vector space \mathbb{C}^d of dimension d , M_d is a point and $G_d = GL_d(\mathbb{C})$. Therefore $\mathcal{H}_d = H_{GL_d(\mathbb{C})}^*(M_d) = \mathbb{Q}[x_{1,d}, \dots, x_{d,d}]^{S_d}$ is the algebra of symmetric polynomials in variables $x_{1,d}, \dots, x_{d,d}$. It is possible to talk about the geometric interpretation of these variables. They can be treated as the first Chern classes of the tautological bundles over the classifying space of G_d . For details see e.g. [10].

The COHA \mathcal{H} for quiver A_1 is described in [5]. It is the infinite exterior algebra generated by odd elements $\phi_0, \phi_1, \phi_2 \dots$ with wedge \wedge as its product. Generators $(\phi_i)_{i \geq 0}$ correspond to the additive generators $(x_{1,1}^i)_{i \geq 0}$ of $\mathcal{H}_1 = \mathbb{Q}[x_{1,1}]$. A monomial in the exterior algebra

$$\phi_{k_1} \wedge \dots \wedge \phi_{k_d} \in \mathcal{H}_d, \quad 0 \leq k_1 < \dots < k_d$$

corresponds to the Schur symmetric polynomial $s_\lambda(x_{1,d}, \dots, x_{d,d})$, where $\lambda = (\lambda_d, \dots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \dots, k_1)$ is a partition.

Let $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \dots \wedge \phi_{k_d}$ with index $\mathbf{k} = (k_1, \dots, k_d)$, $0 \leq k_1 < \dots < k_d$. Denote by $\mathbf{k}(\lambda)$ the index related to the partition λ and by $\lambda(\mathbf{k})$ the partition related to the basis index \mathbf{k} . Then we have $\Phi_{\mathbf{k}(\lambda)} = s_{\lambda(\mathbf{k})}$.

2.3. Stable framed representations. Fix a dimension vector $\mathbf{n} = (n_i)_{i=1}^N$. A *framed representation* of Q of dimension vector γ is a pair (V, f) , where V is an ordinary representation of Q of dimension γ and $f = (f_i)_{i=1}^N$ is a collection of linear maps from \mathbb{C}^{n_i} to V_i . The set of framed representations of dimension vector γ with framed structure dimension vector \mathbf{n} is denoted by $\hat{M}_{\gamma, \mathbf{n}}$. It carries a natural gauge group G_γ -action. See e.g. [7].

For the notion of stable framed representation of a quiver, see e.g. [6] (more general framework of triangulated categories can be found in [9]). We focus on the trivial

stability condition. In this case, a framed representation is called *stable* if there is no proper (ordinary) subrepresentation of V which contains the image of f . The set of stable framed representations of dimension vector γ with framed structure dimension vector \mathbf{n} is denoted by $\hat{M}_{\gamma, \mathbf{n}}^{st}$. The gauge group G_γ of $M_{\gamma, \mathbf{n}}$ induces a G_γ -action on $\hat{M}_{\gamma, \mathbf{n}}^{st}$. The stack of stable framed representations $[\hat{M}_{\gamma, \mathbf{n}}^{st}/G_\gamma]$ is in fact a smooth projective scheme. We denote it by $\mathcal{M}_{\gamma, \mathbf{n}}^{st}$ and call it *the smooth model* of quiver Q with dimension γ and framed structure \mathbf{n} .

The pullback-pushforward construction is applied to the cohomology of the scheme of stable framed representations. This construction leads to two representations of COHA for the quiver Q which we describe below.

Fix two dimension vectors γ_1 and γ_2 . Set $\gamma = \gamma_1 + \gamma_2$. Consider the scheme consisting of diagrams

$$\mathcal{M}_{\gamma_2, \gamma, \mathbf{n}}^{st} := \{ 0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0 \}, \quad (2.1)$$

$\begin{array}{c} \uparrow f \\ \mathbb{C}^{\mathbf{n}} \end{array} \quad \begin{array}{c} \nearrow f_2 \end{array}$

where $E_1 \in M_{\gamma_1}$, $(E, f) \in \mathcal{M}_{\gamma, \mathbf{n}}^{st}$, $(E_2, f_2) \in \mathcal{M}_{\gamma_2, \mathbf{n}}^{st}$. $f : \mathbb{C}^{\mathbf{n}} \rightarrow E$ and $f_2 : \mathbb{C}^{\mathbf{n}} \rightarrow E_2$ are the framed structures attached to E and E_2 respectively. The subgroup of the automorphism group of E which preserves the embedding of E_1 is denoted by $P_{\gamma_1, \gamma, \mathbf{n}}$. It plays the role of the automorphism group of $\mathcal{M}_{\gamma_2, \gamma, \mathbf{n}}^{st}$. The natural projections from the diagram to its components give the following diagram:

$$\begin{array}{ccc} & \mathcal{M}_{\gamma, \mathbf{n}}^{st} & \\ & \uparrow p & \\ & [\hat{M}_{\gamma_2, \gamma, \mathbf{n}}^{st}/P_{\gamma_2, \gamma, \mathbf{n}}] & \\ p_1 \swarrow & & \searrow p_2 \\ [M_{\gamma_1}/G_{\gamma_1}] & & \mathcal{M}_{\gamma_2, \mathbf{n}}^{st} \end{array} \quad (2.2)$$

The map $p_*(p_1^*(\phi_1) \cup p_2^*(\varphi_2))$ defines a morphism from $H^*(\mathcal{M}_{\gamma_2, \mathbf{n}}^{st})$ to $H^*(\mathcal{M}_{\gamma, \mathbf{n}}^{st})$ for $\phi_1 \in \mathcal{H}_{\gamma_1}$ and $\varphi_2 \in H^*(\mathcal{M}_{\gamma_2, \mathbf{n}}^{st})$. This morphism induces a representation of $\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma$ on $\bigoplus_\gamma H^*(\mathcal{M}_{\gamma, \mathbf{n}}^{st})$. It is called *the increasing representation* of COHA for the quiver Q , and denoted by $R_{\mathbf{n}}^+$. Similarly, the map $(p_2)_*(p_1^*(\phi_1) \cup p^*(\varphi))$ for $\phi_1 \in \mathcal{H}_{\gamma_1}$ and $\varphi \in H^*(\mathcal{M}_{\gamma, \mathbf{n}}^{st})$ gives *the decreasing representation* $R_{\mathbf{n}}^-$ on the cohomology of the smooth model. In order to have well-defined representations one needs to show that p and p_2 are proper morphisms. For A_1 -case the properness is obvious (see Section 2.5 below).

2.4. A_1 -case. Let n be the framed structure dimension. A framed representation (\mathbb{C}^d, f) of A_1 -quiver is stable if and only if $f : \mathbb{C}^n \rightarrow \mathbb{C}^d$ is surjective. Thus the stable framed moduli space $\mathcal{M}_{d,n}^{st}$ is the Grassmannian (of quotient spaces) $Gr(d, n)$ for $0 \leq d \leq n$, and empty for $d > n$.

It is well known (see e.g. [3], p.161) that the cohomology of full flag variety $Fl(n)$ is isomorphic to $R(n) = \mathbb{Q}[x_1, \dots, x_n]/(e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n))$, where $e_i(x_1, \dots, x_n)$ represents the i -th elementary symmetric polynomial. The cohomology of Grassmannian $Gr(d, n)$ is a subalgebra of $R(n)$ which is generated by Schur polynomials in variables x_1, \dots, x_d . Thus we can use $s_\lambda(x_1, \dots, x_d)$ to represent classes in $H^*(Gr(d, n))$. There is a natural projection $\pi : Fl(n) \rightarrow Gr(d, n)$. By abuse of notations, we use the same symbol x_i to denote the classes in $Gr(d, n)$ whose pullback $\pi^*(x_i)$ is $x_i \in H^*(Fl(n))$.

Classes in $H^*(Gr(d, n))$ have an alternative presentation.

Lemma 2.1. *In $H^*(Gr(d, n))$, $s_\lambda(x_1, \dots, x_d) = (-1)^{|\lambda|} s_{\lambda'}(x_{d+1}, \dots, x_n)$, where λ' is the transpose partition of λ .*

Proof: The above identity can be easily deduced from the identity $\prod_{i=1}^d \frac{1}{1-x_i t} = \prod_{i=d+1}^n (1-x_i t)$ (see e.g. [3], p.163) in the ring $R(n)[t]$.

Since

$$\prod_{i=1}^d \frac{1}{1-x_i t} = \sum_{r \geq 0} h_r(x_1, \dots, x_d) t^r \quad (2.3)$$

and

$$\prod_{i=d+1}^n (1-x_i t) = \sum_{r \geq 0} e_r(x_{d+1}, \dots, x_n) (-t)^r \quad (2.4)$$

where h_r (resp. e_r) stands for the r -th complete symmetric polynomial (resp. elementary symmetric polynomial), we have

$$h_r(x_1, \dots, x_d) = (-1)^r e_r(x_{d+1}, \dots, x_n), \quad r \geq 0. \quad (2.5)$$

By Jacobi-Trudi identity,

$$\begin{aligned} s_{\lambda'}(x_1, \dots, x_d) &= \det(e_{\lambda_i - i + j}(x_1, \dots, x_d)) \\ &= \det((-1)^{\lambda_i - i + j} h_{\lambda_i - i + j}(x_{d+1}, \dots, x_n)) \\ &= (-1)^{|\lambda|} \det(h_{\lambda_i - i + j}(x_{d+1}, \dots, x_n)) \\ &= (-1)^{|\lambda|} s_\lambda(x_{d+1}, \dots, x_n). \end{aligned}$$

The third identity comes from the fact that

$$\begin{aligned}
\det(h_{\lambda_i-i+j} t^{\lambda_i-i+j}) &= \sum_{\omega} \sum_{i=1}^n (-1)^{\omega} h_{\lambda_i-i+\omega(i)} t^{\lambda_i-i+\omega(i)} \\
&= \sum_{\omega} t^{\sum_{i=1}^n \lambda_i-i+\omega(i)} \sum_{i=1}^n (-1)^{\omega} h_{\lambda_i-i+\omega(i)} \\
&= \sum_{\omega} t^{|\lambda|} \sum_{i=1}^n (-1)^{\omega} h_{\lambda_i-i+\omega(i)} \\
&= t^{|\lambda|} \det(h_{\lambda_i-i+j}).
\end{aligned}$$

□

In the following we will use this “transpose” presentation to do some computations.

2.5. Two representations of A_1 -COHA. The scheme $[M_{d_2,d,n}^{st}/P_{d_2,d,n}]$ in A_1 -quiver case is isomorphic as a scheme to the two-step flag variety $F_{d_2,d,n}$, which is variety of the flags $\{\mathbb{C}^n \twoheadrightarrow \mathbb{C}^d \twoheadrightarrow \mathbb{C}^{d_2}\}$. Let ϕ_i be a generator of \mathcal{H}_1 , and s_{λ} be the Schur polynomial considered as an element of the cohomology of the Grassmannian $H^*(Gr(d_2, n))$ whose partition is λ . In this case, p is the obvious projection from $F_{d_2,d,n}$ to $Gr(d, n)$ and p_2 is the obvious projection from $F_{d_2,d,n}$ to $Gr(d_2, n)$. Therefore both p and p_2 are proper morphisms of stacks (which are in fact schemes), and the increasing and decreasing representations introduced in Section 2.3 are well defined.

Now we want to compute the increasing representation by the formula $p_*(p_1^*(\phi_i) \cup p_2^*(s_{\lambda}))$. Note that in this case, $d_1 = 1$. Recall that ϕ_i represents the polynomial $\phi_i(x_{1,1}) = x_{1,1}^i$. Using the geometric interpretation, $x_{1,1}^i$ is treated as the first Chern class of the tautological line bundle $\mathcal{O}(-i)$ over the classifying space of G_1 . $\mathcal{O}(-i)$ will be pulled back through p_1 to the line bundle over $F_{d_2,d,n}$ associated to the corresponding character of G_{d_1} when treating G_{d_1} as a subquotient of $P_{d_2,d,n}$. Hence $p_1^*(\phi_i)$ will be the first Chern class of the line bundle described above, which is $\phi_i(x_{d_2+1}) = x_{d_2+1}^i$.

As homogenous spaces, $Gr(d, n) \approx GL_n(\mathbb{C})/P_{d,n}$, $Gr(d_2, n) \approx GL_n(\mathbb{C})/P_{d_2,n}$ and $F(d_2, d, n) \approx GL_n(\mathbb{C})/P_{d_2,d,n}$. We use the formula in [1] to compute the pushforward.

Theorem 2.2. [1]. *Let G be a connected reductive algebraic group over \mathbb{C} and B a Borel subgroup. Choose a maximal torus $T \subset B$ with Weyl group W . The set of all positive roots of the root system of (G, T) is denoted by Δ^+ . Let $P \supset B$ be a parabolic subgroup of G , with the set of positive roots $\Delta^+(P)$ and Weyl group W_P . Let L_{α} be the complex line bundle over G/B which is associated to the root α . The Gysin homomorphism $f_* : H^*(G/B) \rightarrow$*

$H^*(G/P)$ is given by

$$f_*(p) = \sum_{w \in W/W_P} w \cdot \frac{p}{\prod_{\alpha \in \Delta^+ \setminus \Delta^+(P)} c_1(L_\alpha)}. \quad (2.6)$$

Applying Thm 2.2, for $s_\lambda \in H^*(Gr(d_2, n))$,

$$(\phi_i^+ \cdot s_\lambda)(x_1, \dots, x_{d_2+1}) = \sum_{i_1 < \dots < i_{d_2}} \frac{s_\lambda(x_{i_1}, \dots, x_{i_{d_2}}) \phi_i(x_{i_{d_2}+1})}{\prod_{j=1}^{d_2} (x_{i_j} - x_{i_{d_2}+1})}. \quad (2.7)$$

Similarly, the formula of the decreasing actions is

$$(\phi_i^- \cdot s_\lambda)(x_1, \dots, x_{d_2-1}) = \sum_{i_1 < \dots < i_{d_2}} \frac{s_\lambda(x_{i_1}, \dots, x_{i_{d_2}}) \phi_i(x_{i_{d_2}})}{\prod_{j=d_2+1}^n (x_{i_{d_2}} - x_j)}. \quad (2.8)$$

Remark 2.3. In Formula (2.8), variables x_i for $i > d_2 - 1$ appear on the right side, which do not belong to the variables on the left side. This is not a contradiction because of the formula $s_\lambda(x_1, \dots, x_d) = (-1)^{|\lambda|} s_{\lambda'}(x_{d+1}, \dots, x_n)$ by Lemma 2.1. More details will be discussed in the following section.

Remark 2.4. The construction above actually only defines an increasing operator $\phi_{i,d}^+$ from $H^*(Gr(d, n))$ to $H^*(Gr(d+1, n))$ and an decreasing operator $\phi_{i,d}^-$ from $H^*(Gr(d, n))$ to $H^*(Gr(d-1, n))$. The increasing operator we need is $\phi_i^+ = \sum_{d=0}^n \phi_{i,d}^+$. The decreasing operator we need is $\phi_i^- = \sum_{d=0}^n \phi_{i,d}^-$. We can then define the *twisted decreasing operator* by $\hat{\phi}_i^- = \sum_{d=0}^n (-1)^{d-1} \phi_{i,d}^-$. We call the representation formed by these operators the *twisted decreasing representation* and denote it by \hat{R}_n^- .

3. INCREASING AND DECREASING OPERATORS

3.1. Increasing operators. The key result of this subsection is adapted from [2].

Proposition 3.1. [2]. *The increasing representation structure is induced by the open embedding $j : \hat{M}_{d,n}^{st} \rightarrow \hat{M}_{d,n}$. The induced map $j^* : \mathcal{H} \rightarrow R_n^+$ is \mathcal{H} -linear and surjective. The kernel of j^* equals $\sum_{p \geq 0, q > 0} \mathcal{H}_p \wedge (e_q^n \cup \mathcal{H}_q)$, where $e_q = \prod_{i=1}^d x_i$.*

Proof: In [2], the similar result for $n = 1$ is proved. It can be easily generalized to $n > 1$ case for A_1 -quiver. \square

The next lemma follows immediately from the definition of Schur polynomials.

Lemma 3.2. $s_{(\lambda_d+1, \lambda_{d-1}+1, \dots, \lambda_1+1)} = e_d s_\lambda$ for $s_\lambda \in \mathbb{Q}[x_1, \dots, x_d]^{S_d}$ and $e_d = \prod_{i=1}^d x_i$. Thus $e_d^n \cup \Phi_{\mathbf{k}} = \Phi_{\mathbf{k}+\mathbf{n}}$ for $\Phi_{\mathbf{k}} \in \mathcal{H}_d$, and $\mathbf{n} = (n, n, \dots, n)$.

Finally, we come to the result, whose proof is straightforward.

Proposition 3.3. *The increasing representation R_n^+ is a quotient of $\mathcal{H} = \bigwedge^*(\mathcal{H}_1)$ whose kernel is the subalgebra generated by $\{\phi_i\}_{i \geq n}$. Thus R_n^+ is isomorphic to $\bigwedge^*(V(n))$ where $V(n)$ is the linear space spanned by $\phi_0, \dots, \phi_{n-1}$ and the action is given by wedge product from left. Then $\{\phi_{k_1} \wedge \dots \wedge \phi_{k_d}\}_{k_1 < \dots < k_d, 0 \leq d \leq n-1}$ form a basis of R_n^+ .*

3.1.1. *Two presentations of classes in the cohomology of Grassmannian.* Proposition 3.3 implies that we can use the notations introduced in section 2.2 to represent cohomology classes of Grassmannians, as well as those in COHA, since they share the same product structure. Thus in $H^*(Gr(d, n))$, $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \dots \wedge \phi_{k_d}(x_1, \dots, x_d)$ with index $\mathbf{k} = (k_1, \dots, k_d)$ can represent the Schur polynomial $s_{\lambda(\mathbf{k})}(x_{1,d}, \dots, x_{d,d})$, where $0 \leq k_1 < \dots < k_d \leq n-1$ and $\lambda = (\lambda_d, \dots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \dots, k_1)$ is a partition of length $\leq n$.

Let λ' be the transpose partition of λ , and $\mathbf{k}' = \mathbf{k}(\lambda')$. By Lemma 2.1, $\Phi_{\mathbf{k}}(x_1, \dots, x_d) = (-1)^{|\lambda|} \Phi_{\mathbf{k}'}(x_{d+1}, \dots, x_n)$. $\Phi_{\mathbf{k}}$ is called *the ordinary presentation* of the correspondent class s_{λ} , and $(-1)^{|\lambda|} \Phi_{\mathbf{k}'}$ is called *the transpose presentation*.

3.2. **Decreasing operators.** Our goal is to understand the decreasing representation using the basis $\{\Phi_{\mathbf{k}}\}_{\mathbf{k}}$ of R_n^+ . From Section 3.1.1, the equation (2.8) can be rewritten as

$$\begin{aligned} (\phi_i^- \cdot \Phi_{\mathbf{k}})(x_1, \dots, x_{d_2-1}) &= \sum_{i_1 < \dots < i_{d_2}} \frac{\Phi_{\mathbf{k}}(x_{i_1}, \dots, x_{i_{d_2}}) \phi_i(x_{i_{d_2}})}{\prod_{j=d_2+1}^n (x_{i_{d_2}} - x_j)} \\ &= (-1)^{|\lambda(\mathbf{k})|} \sum_{i_{d_2+1} < \dots < i_n} \frac{\Phi_{\mathbf{k}'}(x_{i_{d_2+1}}, \dots, x_{i_n}) \phi_i(x_{i_{d_2}})}{\prod_{j=d_2+1}^n (x_{i_{d_2}} - x_{i_j})} \\ &= (-1)^{|\lambda|+n-d_2} (\phi_i^+ \cdot \Phi_{\mathbf{k}'})(x_{d_2}, \dots, x_n). \end{aligned} \quad (3.1)$$

This formula suggests an algorithm. Start from an ordinary presentation of a class $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \dots \wedge \phi_{k_d}$ in $H^*(Gr(d, n))$, where $\mathbf{k} = (k_1, \dots, k_d)$, and $0 \leq k_1 < \dots < k_d \leq n-1$. First we change $\Phi_{\mathbf{k}}(x_1, \dots, x_d)$ to $(-1)^{|\lambda(\mathbf{k})|} \Phi_{\mathbf{k}'}(x_{d+1}, \dots, x_n)$ by Lemma 2.1. Then apply ϕ_i^- to $\Phi_{\mathbf{k}'}$ using formula (3.1) and Proposition 3.3. Finally change the result back to the ordinary presentation.

We need the following lemma to help us to do these transformations.

Lemma 3.4. *If ϕ_r appears in $\Phi_{\mathbf{k}'(\lambda)}$, ϕ_{n-r-1} will not appear in $\Phi_{\mathbf{k}(\lambda)}$. On the other hand, if ϕ_r doesn't appear in $\Phi_{\mathbf{k}'(\lambda)}$, ϕ_{n-r-1} will appear in $\Phi_{\mathbf{k}(\lambda)}$.*

Proof: From Section 3.3, $\lambda = (\lambda_d, \dots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \dots, k_1)$ is a partition of length $\leq n$. The transpose partition is defined by $\lambda'_j = \#\{\lambda_i \geq n - d + 1 - j\}$ for

$1 \leq j \leq n - d$. Thus we have

$$\lambda_{d-i+1} = \begin{cases} n - d & \text{if } 1 \leq i \leq \lambda'_1, \\ n - d - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ for } 1 \leq j \leq n - d - 1, \\ 0 & \text{if } \lambda'_{n-d} + 1 \leq i \leq d. \end{cases} \quad (3.2)$$

From $\lambda = (\lambda_d, \dots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \dots, k_1)$, it immediately implies

$$k_{d-i+1} = \begin{cases} n - i & \text{if } 1 \leq i \leq \lambda'_1, \\ n - i - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ for } 1 \leq j \leq n - d - 1, \\ d - i & \text{if } \lambda'_{n-d} + 1 \leq i \leq d. \end{cases} \quad (3.3)$$

Then $n - k'_{j+1} = n - j - \lambda'_{j+1} \leq k_{d-i+1} = n - i - j \leq n - j - \lambda'_j - 1 = n - k'_j - 2$ if $\lambda'_j + 1 \leq i \leq \lambda'_{j+1}$ for $1 \leq j \leq n - d - 1$, or $0 = d - d \leq k_{d-i+1} = d - i \leq d - \lambda'_{n-d} - 1 = n - k'_{n-d} - 2$ if $\lambda'_{n-d} + 1 \leq i \leq d$, or $n - k'_1 = n - \lambda'_1 \leq k_d = n - i \leq n - 1$. Therefore k_{d-i+1} would run over all integers between $n - k'_{j+1}$ and $n - k'_j - 2$, or between 0 and $n - k'_{n-d} - 2$, or between $n - k'_1$ and $n - 1$. If ϕ_r doesn't appear in $\Phi_{\mathbf{k}'(\lambda)}$, there are three cases. If $k'_s < r < k'_{s+1}$ for $1 \leq s \leq n - d - 1$, $n - k'_{s+1} \leq n - r - 1 \leq n - k'_s - 2$. If $k'_{n-d} < r \leq d$, $0 \leq n - r - 1 \leq n - k'_{n-d} - 2$. If $0 \leq r < k'_1$, $n - k'_1 \leq n - r - 1 \leq n - 1$. This means that there exists some $1 \leq i \leq d$ such that $k_{d-i+1} = n - r - 1$.

If ϕ_r appear in $\Phi_{\mathbf{k}'(\lambda)} = \phi_{k'_1} \wedge \dots \wedge \phi_{k'_{n-d}}$, let $r = k'_s$. Then k_{d-i+1} can never be $n - k'_s - 1 = n - r - 1$ for $1 \leq i \leq d$.

□

Definition 3.5. We introduce the *right partial derivative operator* $\partial_i^R : \bigwedge^*(V(n)) \rightarrow \bigwedge^*(V(n))$ to state the following proposition. For $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \dots \wedge \phi_{k_d}$, if ϕ_i appears in $\Phi_{\mathbf{k}}$, $\partial_i^R(\Phi_{\mathbf{k}}) = (-1)^{d-i} \phi_{k_1} \wedge \dots \wedge \hat{\phi}_i \wedge \dots \wedge \phi_{k_d}$. If ϕ_i does not appear in $\Phi_{\mathbf{k}}$, $\partial_i^R(\Phi_{\mathbf{k}}) = 0$. The *left partial derivative operator* $\partial_i^L : \bigwedge^*(V(n)) \rightarrow \bigwedge^*(V(n))$ is defined in the similar way. If ϕ_i appears in $\Phi_{\mathbf{k}}$, $\partial_i^L(\Phi_{\mathbf{k}}) = (-1)^{i-1} \phi_{k_1} \wedge \dots \wedge \hat{\phi}_i \wedge \dots \wedge \phi_{k_d}$. If ϕ_i does not appear in $\Phi_{\mathbf{k}}$, $\partial_i^L(\Phi_{\mathbf{k}}) = 0$. It is easy to see that $\partial_i^R = (-1)^{d-1} \partial_i^L$ on $\bigwedge^d(V(n))$.

Proposition 3.6. *The decreasing operators are the right partial derivative operators on $\bigwedge^*(V(n))$: $\phi_r^- \cdot \Phi_{\mathbf{k}} = \partial_{n-r-1}^R(\Phi_{\mathbf{k}})$.*

Proof: What we want is to compute $\phi_r^- \cdot \Phi_{\mathbf{k}}$. Based on formula (3.1), we have

$$\begin{aligned} (\phi_r^- \cdot \Phi_{\mathbf{k}})(x_1, \dots, x_{d-1}) &= (-1)^{|\lambda|+n-d} (\phi_r^+ \cdot \Phi_{\mathbf{k}'}) (x_d, \dots, x_n) \\ &= (-1)^{|\lambda|+n-d} (\phi_r \wedge \phi_{k'_1} \wedge \dots \wedge \phi_{k'_{n-d}})(x_d, \dots, x_n). \end{aligned} \quad (3.4)$$

If ϕ_{n-r-1} is not in the $\Phi_{\mathbf{k}}$, ϕ_r will appear in $\Phi_{\mathbf{k}'}$. Thus $\phi_r^- \cdot \Phi_{\mathbf{k}}(x_1, \dots, x_{d-1}) = (\phi_r \wedge \phi_{k'_1} \wedge \dots \wedge \phi_r \wedge \dots \wedge \phi_{k'_{n-d}})(x_d, \dots, x_n) = 0$.

If ϕ_{n-r-1} appears in $\Phi_{\mathbf{k}} = \phi_{k_1} \wedge \dots \wedge \phi_{k_d}$, ϕ_r won't be in $\Phi_{\mathbf{k}'} = \phi_{k'_1} \wedge \dots \wedge \phi_{k'_{n-d}}$. Assume $k'_s < r < k'_{s+1}$. We have

$$\phi_r \wedge \phi_{k'_1} \wedge \dots \wedge \phi_{k'_{n-d}} = (-1)^s \phi_{k'_1} \wedge \dots \wedge \phi_{k'_s} \wedge \phi_r \wedge \phi_{k'_{s+1}} \wedge \dots \wedge \phi_{k'_{n-d}}. \quad (3.5)$$

We have to change this back to the ordinary presentation. First, let's find the partition associated to this polynomial. The index $\mathbf{l}' = (l'_1, \dots, l'_{n-d+1})$ is given by

$$l'_i = \begin{cases} k'_{i-1} & s+2 \leq i \leq n-d+1, \\ r & i = s+1, \\ k'_i & 1 \leq i \leq s. \end{cases} \quad (3.6)$$

Then the new partition $\mu' = (\mu'_{n-d+1}, \dots, \mu'_1)$ is given by

$$\mu'_i = \begin{cases} \lambda'_{i-1} - 1 & s+2 \leq i \leq n-d+1, \\ r-s & i = s+1, \\ \lambda'_i & 1 \leq i \leq s. \end{cases} \quad (3.7)$$

Next step is to recover the partition μ from its transpose μ' . From the definition of transpose partition, $\mu'_j = \#\{\mu_i \geq n-d+2-j\}$ for $1 \leq j \leq n-d-1$. Then

$$\mu_{d-i} = \begin{cases} n-d-j & \text{if } \lambda'_j \leq i \leq \lambda'_{j+1} - 1 \text{ and } s+1 \leq j \leq n-d-1, \\ n-d-s & \text{if } r-s+1 \leq i \leq \lambda'_{s+1} - 1, \\ n-d+1-s & \text{if } \lambda'_s + 1 \leq i \leq r-s, \\ n-d+1-j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ and } 2 \leq j \leq s-1, \\ n-d+1 & \text{if } 1 \leq i \leq \lambda'_1. \end{cases} \quad (3.8)$$

By comparing it with

$$\lambda_{d-i+1} = \begin{cases} n-d-j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ and } 2 \leq j \leq n-d-1, \\ n-d & \text{if } 1 \leq i \leq \lambda'_1, \end{cases} \quad (3.9)$$

we notice that $\mu_i = \lambda_{i+1} + 1$ for $d-r+s \leq i \leq d-1$ and $\mu_i = \lambda_i$ for $1 \leq i \leq d-r+s-1$.

Therefore, since $l_i = \mu_i + i - 1$ for $1 \leq i \leq d-1$ and $k_j = \lambda_j + j - 1$ for $1 \leq j \leq d$, it is easy to see that $l_i = k_{i+1}$ for $d-r+s \leq i \leq d-1$ and $l_i = k_i$ for $1 \leq i \leq d-r+s-1$. Thus the resulted presentation is $(-1)^{n-d+s+|\lambda|+|\mu|} \phi_{k_1} \wedge \dots \wedge \hat{\phi}_{n-r-1} \wedge \dots \wedge \phi_{k_d} = (-1)^{r+s} \phi_{k_1} \wedge \dots \wedge \hat{\phi}_{n-r-1} \wedge \dots \wedge \phi_{k_d} = \partial_{n-r-1}^R(\Phi_{\mathbf{k}})$, which is $\Phi_{\mathbf{k}}$ applied by the right partial derivative of ϕ_{n-r-1} . If $r < k'_1$ or $r > k'_{n-d}$, the similar process will lead to the same result. \square

3.3. Twisted decreasing operators. From the above computations, it is obvious to have the following proposition about the twisted decreasing operators.

Proposition 3.7. *The twisted decreasing operators are the left partial derivative operators on $\bigwedge^*(V(n))$: $\hat{\phi}_r^- \cdot \Phi_k = \partial_{n-r-1}^L(\Phi_k)$.*

4. THE DOUBLE OF REPRESENTATIONS

4.1. The double of untwisted representations. Let $V(n)$ be the n -dimensional vector space spanned by $\{\phi_i\}_{i=0}^{n-1}$. The increasing and decreasing representations can be realized as creation operators $\{\alpha_i^+\}_{i=0}^{n-1}$ and annihilation operators $\{\alpha_i^-\}_{i=0}^{n-1}$ on $\bigwedge^*(V(n))$. Here $\alpha_i^+ = \phi_i^+$ is the left wedge product, and $\alpha_i^- = \phi_{n-i-1}^-$ is the right partial derivative ∂_i^R .

Define $H = [\alpha_0^+, \alpha_0^-]$ and the following operators for $0 \leq i \leq n-1$:

$$T_i = \frac{\alpha_i^+ + [H, \alpha_i^+]/2}{2}, \quad S_i = \frac{\alpha_i^- - [H, \alpha_i^-]/2}{2}.$$

Then define the following operators

$$\begin{aligned} E_0 &= -\frac{\alpha_0^- + [H, \alpha_0^-]/2}{2}, \quad F_0 = \frac{\alpha_0^+ - [H, \alpha_0^+]/2}{2}, \\ E_1 &= S_0, \quad F_1 = T_0, \\ E_i &= [T_{i-2}, S_{i-1}], \quad F_i = [T_{i-1}, S_{i-2}], \quad \text{for } 2 \leq i \leq n, \\ H_i &= [E_i, F_i], \quad \text{for } 0 \leq i \leq n. \end{aligned}$$

In the following, let P_k be an arbitrary degree k monomial in $\bigwedge^*(V(n))$. Denote by R_i^j the operator which change the factor ϕ_i in P_k to ϕ_j .

Lemma 4.1. *For $2 \leq i \leq n$,*

- (1) $H(P_k) = (-1)^{k-1} P_k$.
- (2) $E_0(P_k) = -\partial_0^R(P_k)$ if k is even, and ϕ_0 is included in P_k . Otherwise it's 0.
- (3) $F_0(P_k) = \phi_0 \wedge P_k$ if k is odd, and ϕ_0 is NOT included in P_k . Otherwise it's 0.
- (4) $E_1(P_k) = \partial_0^R(P_k)$ if k is odd, and ϕ_0 is included in P_k . Otherwise it's 0.
- (5) $F_1(P_k) = \phi_0 \wedge P_k$ if k is even, and ϕ_0 is NOT included in P_k . Otherwise it's 0.
- (6) $S_{i-1}(P_k) = \partial_{i-1}^R(P_k)$ if k is odd, and ϕ_{i-1} is included in P_k . Otherwise it's 0.
- (7) $T_{i-1}(P_k) = \phi_{i-1} \wedge P_k$ if k is even, and ϕ_{i-1} is NOT included in P_k . Otherwise it's 0.
- (8) $E_i(P_k) = R_{i-1}^{i-2}(P_k)$ if ϕ_{i-1} is included in P_k and ϕ_{i-2} is NOT. Otherwise it's 0.
- (9) $F_i(P_k) = R_{i-2}^{i-1}(P_k)$ if ϕ_{i-2} is included in P_k and ϕ_{i-1} is NOT. Otherwise it's 0.

$$\begin{aligned}
(10) \quad H_0(P_k) &= \begin{cases} -P_k & k \text{ is even and } \phi_0 \text{ is included in } P_k \\ P_k & k \text{ is odd and } \phi_0 \text{ is NOT included in } P_k. \\ 0 & \text{otherwise} \end{cases} \\
(11) \quad H_1(P_k) &= \begin{cases} P_k & k \text{ is even and } \phi_0 \text{ is NOT included in } P_k \\ -P_k & k \text{ is odd and } \phi_0 \text{ is included in } P_k \\ 0 & \text{otherwise} \end{cases} . \\
(12) \quad H_i(P_k) &= \begin{cases} -P_k & \phi_{i-1} \text{ is included in } P_k \text{ and } \phi_{i-2} \text{ is NOT included} \\ P_k & \phi_{i-2} \text{ is included in } P_k \text{ and } \phi_{i-1} \text{ is NOT included.} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Proof: The proof of the lemma is straightforward. \square

The main theorem below implies that the combination of two representations R_n^+ and R_n^- of A_1 -COHA forms an D_{n+1} -Lie algebra.

Theorem 4.2. *The above operators satisfy the Serre relations for $0 \leq i, j \leq n$:*

- (1) $[H_i, H_j] = 0$,
- (2) $[E_i, F_j] = \delta_{ij}H_i$,
- (3) $[H_i, E_j] = a_{ji}E_j, \quad [H_i, F_j] = -a_{ji}F_j$,
- (4) $(adE_i)^{-a_{ji}+1}(E_j) = 0$, if $i \neq j$,
- (5) $(adF_i)^{-a_{ji}+1}(F_j) = 0$, if $i \neq j$,

where (a_{ij}) is the Cartan matrix for D_{n+1} -Lie algebras.

Proof: The first statement holds since each H_i is diagonal by Lemma 4.1. The second is due to the definition of H_i for $\delta_{ij} = 1$. For the other relations, we need to check the following relations, which can be easily solved by Lemma 4.1:

- (1) $a_{ii} = 2$, for $0 \leq i \leq n$,
- (2) $a_{21} = a_{20} = a_{12} = a_{02} = a_{i-1,i} = a_{i,i-1} = -1$ for $3 \leq i \leq n$,
- (3) $a_{10} = a_{01} = a_{0,i} = a_{i,0} = a_{1,i} = a_{i,1} = a_{i,j} = a_{j,i} = 0$, for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$
- (4) $[E_0, F_1] = [E_1, F_0] = [E_0, F_j] = [E_1, F_j] = [E_i, F_0] = [E_i, F_1] = [E_i, F_j] = 0$, for $2 \leq i \neq j \leq n$,
- (5) $[E_2, [E_2, E_0]] = [E_0, [E_0, E_2]] = [F_2, [F_2, F_0]] = [F_0, [F_0, F_2]] = 0$,
- (6) $[E_{i-1}, [E_{i-1}, E_i]] = [E_i, [E_i, E_{i-1}]] = [F_{i-1}, [F_{i-1}, F_i]] = [F_i, [F_i, F_{i-1}]] = 0$, for $2 \leq i \leq n$,

(7) $[E_0, E_1] = [E_0, E_i] = [E_1, E_i] = [E_i, E_j] = 0$ for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$,

(8) $[F_0, F_1] = [F_0, F_i] = [F_1, F_i] = [F_i, F_j] = 0$ for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$.

□

4.2. The double of twisted representations. Use the setting from the previous subsection. Let $\hat{\alpha}_i^- = \hat{\phi}_{n-i-1}^-$ be the left partial derivative ∂_i^L . Now we use creation operators $\{\alpha_i^+\}_{i=0}^{n-1}$ and twisted annihilation operators $\{\hat{\alpha}_i^-\}_{i=0}^{n-1}$ to form representations. These relations show that the double of twisted representations form a finite Clifford algebra.

Theorem 4.3. *Operators $\{\alpha_i^+\}_{i=0}^{n-1}$ and $\{\hat{\alpha}_i^-\}_{i=0}^{n-1}$ satisfy the following relations:*

- (1) $\alpha_i^+ \alpha_j^+ + \alpha_j^+ \alpha_i^+ = 0$,
- (2) $\hat{\alpha}_i^- \hat{\alpha}_j^- + \hat{\alpha}_j^- \hat{\alpha}_i^- = 0$,
- (3) $\alpha_i^+ \hat{\alpha}_j^- + \hat{\alpha}_j^- \alpha_i^+ = \delta_{i,j}$.

Proof: The proof is straightforward by applying the formula in the definitions of the operators to the basis vectors of $\bigwedge^*(V(n))$. □

5. FURTHER DISCUSSIONS

For fixed n , the double of R_n^+ and R_n^- forms D_{n+1} -Lie algebra, and the double of R_n^+ and \hat{R}_n^- forms a finite Clifford algebra. This leads to the following conjecture stated in [9].

Conjecture 5.1. [9] *Full COHA for the quiver A_1 is isomorphic to the infinite Clifford algebra Cl_c with generators ϕ_n^\pm , $n \in \mathbb{Z}$ and the central element c , subject to the standard anticommuting relations between ϕ_n^+ (resp. ϕ_n^-) as well as the relation $\phi_n^+ \phi_m^- + \phi_m^- \phi_n^+ = \delta_{n,m} c$.*

Remark 5.2. As stated in [9], in the case of finite-dimensional representations we have $c \mapsto 0$ and we see two representations of the infinite Grassmann algebra, which are combined in the representations of the orthogonal Lie algebra.

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